JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 14, 285-293 (1966)

Conditionally Invariant Sets and Vector Lyapunov Functions

A. A. KAYANDE AND V. LAKSHMIKANTHAM

Department of Mathematics,
Marathwada University, Aurangabad (Maharashtra), India

Submitted by Richard Bellman

1. As is now well known, one of the most important methods in the study of nonlinear differential equations is the second method of Lyapunov, along with the comparison principle based on the use of single Lyapunov function. Recently Bellman [1], Matrosov [2], and V. Lakshmikantham [3] have shown that using a vector Lyapunov function is advantageous in certain cases. We introduce, in this paper, the concept of a *conditionally invariant set* with respect to a given set and exploit the notion of vector Lyapunov functions to obtain sufficient conditions for conditional stability and boundedness of the same. The example, given at the end, illustrates that these notions hold while the corresponding properties with respect to the given set, need not hold.

2. Let I denote the half-line $0 \leqslant t < \infty$ and R^n denote the Euclidean space of n-dimensions. Consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad (t_0 \in I), \qquad \left({}' \equiv \frac{d}{dt}\right), \tag{2.1}$$

where x and f are n-dimensional vectors and the function $f(t, x)$ is defined and continuous on $I \times R^n$. Let $x(t; t_0, x_0)$ be a solution of (2.1) with $x(t_0; t_0, x_0) = x_0$. Let A and B be any two subsets of R^n such that $A \subset B$.

DEFINITIONS. (1) The set B is said to be *conditionally invariant* with respect to the set A for the differential system (2.1) if $x_0 \in A$ implies that $x(t; t_0, x_0) \subset B$ for all $t \geqslant t_0$.

(2) The set B is said to be *self-invariant* for the differential system (2.1) if $x_0 \in B$ implies that $x(t; t_0, x_0) \subset B$ for all $t \geqslant t_0$.

Let g be a vector of m-dimensions and the function $g(x)$ be defined and continuous on R^n. Define

$$\| g(x) \| = \sqrt{\sum_{i=1}^{m} [g_i(x)]^2}.$$

Let the set A be defined by $\| g(x) \| = 0$. Denote the set of points $\{x : \| g(x) \| < \alpha\}$ and $\{x : \| g(x) \| \leqslant \alpha\}$ by $S(A, \alpha)$ and $\overline{S(A, \alpha)}$, respectively, where α is some positive real number. Let the set $B = \overline{S(A, \alpha)}$ be conditionally invariant with respect to A. Suppose that M_k denotes a manifold of k-dimensions in R^n, $(k \leqslant n)$.

In order to unify our results on conditional stability and boundedness of the conditionally invariant set B with respect to the system (2.1), we state the following definitions. We define $S(B, \epsilon) = S(A, \alpha + \epsilon)$, $\epsilon > 0$ and use below $S(B, \epsilon)$ for clarity.

(i) Given $\epsilon > 0$ and $t_0 \geqslant 0$, there exists a positive function $\delta(t_0, \epsilon, \alpha)$ that is continuous in t_0 for each ϵ, such that

$$x(t; t_0, x_0) \subset S(B, \epsilon), \qquad (t \geqslant t_0),$$

whenever

$$x_0 \in \overline{S(A, \delta)} \cap M_k .$$

(ii) The δ in (i) is independent of t_0.

(iii) Given $\epsilon > 0$, $\gamma \geqslant 0$ and $t_0 \geqslant 0$, there exists a positive number $T = T(t_0, \alpha, \gamma, \epsilon)$ such that

$$x(t; t_0, x_0) \subset S(B, \epsilon), \qquad (t \geqslant t_0 + T),$$

whenever

$$x_0 \in \overline{S(A, \gamma)} \cap M_k .$$

(iv) The T in (iii) is independent of t_0.

(v) Definitions (i) and (iii) hold simultaneously.

(vi) Definitions (ii) and (iv) hold simultaneously.

(vii) Given $\gamma > \alpha$ and $t_0 \geqslant 0$, there exists a positive function $\eta = \eta(t_0, \gamma, \alpha)$ that is continuous in t_0 for each ϵ such that

$$x(t; t_0, x_0) \subset S(B, \eta), \qquad (t \geqslant t_0)$$

whenever

$$x_0 \in \overline{S(A, \gamma)} \cap M_k .$$

(viii) The η in (vii) is independent of t_0.

(ix) For each $\gamma > \alpha$ and $t_0 \geqslant 0$, there exists a positive number β and a positive number $T = T(t_0, \gamma)$ such that

$$x(t; t_0, x_0) \subset S(B, \beta), \qquad (t \geqslant t_0 + T),$$

whenever

$$x_0 \in \overline{S(A, \gamma)} \cap M_k .$$

(x) The T in (ix) is independent of t_0.

(xi) Definitions (vii) and (ix) hold simultaneously.

(xii) Definitions (viii) and (x) hold simultaneously.

REMARKS. (1) We observe that the set B need not be self-invariant.

(2) If $\alpha = 0$ these definitions reduce to the corresponding definitions for conditional stability and boundedness of the self-invariant set A.

(3) If $M_k = R^n$ these definitions reduce to the corresponding definitions for stability and boundedness of the conditionally invariant set B.

(4) If $\alpha = 0$ and $M_k = R^n$ these definitions reduce to the corresponding definitions for stability and boundedness of the selfinvariant set A.

3. Let W be a vector of N-dimensions and the function $W(t, r)$ be defined and continuous on $I \times R^N$. For each $t \in I$ and for each i, $(i = 1, 2, \cdots, N)$, let $W_i\ (t, r_1, r_2, \cdots, r_N)$ be nondecreasing in $r_1, r_2, \cdots, r_{i-1}, r_{i+1}, \cdots, r_N$. Then it is known [4] that the differential system

$$r' = W(t, r), \qquad r(t_0) = r_0 , \qquad \left(' = \frac{d}{dt}\right), \tag{3.1}$$

has the maximal solution (in the sense of componentwise majorization) existing to the right of t_0.

Let V be a vector of N-dimensions and the function $V(t, x)$ be defined and continuous on $I \times R^n$. We shall assume in the sequel that any inequality specified in terms of any two vectors of the same dimension implies that the same inequality holds for each corresponding component of the given vectors. If 0 denotes the N-dimensional zero vector, assume that

$$0 \leqslant V(t, x). \tag{3.2}$$

Suppose further that $V(t, x)$ satisfies for each t, a Lipschitz's condition in x locally. Define

$$V^*(t, x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)]. \tag{3.3}$$

With respect to these functions we state the following lemma which is proved in [3].

LEMMA. *Let the function $V^*(t, x)$ of (3.3) satisfy the inequality*

$$V^*(t, x) \leqslant W(t, V(t, x)),$$

where $W(t, r)$ is the same function, defined earlier having the stated monotonic property. Let $x(t; t_0, x_0)$ be any solution of (2.1) such that $V(t_0, x_0) \leqslant r_0$, then

$$V(t, x(t; t_0, x_0)) \leqslant r(t; t_0, r_0), \qquad (t \geqslant t_0)$$

where $r(t; t_0, r_0)$ is the maximal solution of (3.1).

4. Let $r(t; t_0, r_0)$ be a solution of the differential system (3.1). If $r_0 = 0$, assume that

$$\sum_{i=1}^{N} r_i(t; t_0, 0) \leqslant \beta, \qquad (t \geqslant t_0), \tag{4.1}$$

where $\beta = \beta(t_0)$. Let in general

$$r_0 = [r_{10}, r_{20}, \cdots, r_{k0}, 0, 0, \cdots, 0], \qquad (k < N) \tag{4.2}$$

Corresponding to the definitions (i) to (xii) given in Section 2, if we say that the differential system (3.1) has the property (ia), we mean the following condition is satisfied.

(ia) Given $\epsilon > 0$ and $t_0 \geqslant 0$, there exists a positive function $\delta = \delta(t_0, \epsilon, \beta)$ that is continuous in t_0 for each ϵ such that

$$\sum_{i=1}^{N} r_i(t; t_0, r_0) < \beta + \epsilon, \qquad (t \geqslant t_0),$$

whenever

$$\sum_{i=1}^{N} r_{i0} \leqslant \delta,$$

where r_0 satisfies (4.2) and β is defined by (4.1).

(iia) The β in (4.1) and the δ in (ia) are both independent of t_0. Conditions (iiia) to (xiia) can be defined similarly.

We list below certain assumptions which will be used subsequently.

$$V(t, x) \equiv 0 \qquad \text{if and only if} \qquad x \in A. \tag{4.3}$$

(4.4) The set of points $\{x\}$ defined by $V_i(t, x) \equiv 0$, $(i = k + 1, \cdots, N)$ constitutes a manifold of k-dimensions containing the set A. This manifold will be denoted by M_k.

(4.5) The function $b(r)$ is defined, continuous, and nondecreasing in r, $r \geqslant 0$; $b(r) > 0$ for $r > 0$ and

$$b(\| g(x) \|) \leqslant \sum_{i=1}^{N} V_i(t, x) \qquad \text{for each} \qquad (t, x) \in I \times R^N.$$

(4.6)

$$\sum_{i=1}^{N} V_i(t, x) \to 0 \qquad \text{as} \qquad \| g(x) \| \to 0 \qquad \text{uniformly in} \qquad t \in I.$$

$$b(r) \to \infty \qquad \text{as} \qquad r \to \infty. \tag{4.7}$$

We state the following theorems on conditional stability and boundedness of conditionally Invariant set.

THEOREM 1. *Let the assumptions of the lemma hold together with (4.3), (4.4), (4.5), (4.6), and (4.7). Then*

I. *Condition (ia) $\Rightarrow$ condition (i),*

II. *Condition (iiia) $\Rightarrow$ condition (iii),*

III. *Condition (va) $\Rightarrow$ condition (v).*

Proof. As the lemma holds, we have

$$\sum_{i=1}^{N} V_i(t, x(t; t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t; t_0, r_0), \qquad (t \geqslant t_0) \tag{4.8}$$

whenever

$$\sum_{i=1}^{N} V_i(t_0, x_0) \leqslant \sum_{i=1}^{N} r_{i0}, \tag{4.9}$$

where $x(t; t_0, x_0)$ is any solution of (2.1) and $r(t; t_0, r_0)$ is the maximal solution of (3.1).

Let $r_0 = 0$. Then (4.9) together with (3.2) implies that $\sum_{i=1}^{N} V_i(t_0, x_0) = 0$. But this result yields that $x_0 \in A$ due to (4.3). As (4.5) and (4.7) hold, (4.8) together with (4.1) implies that $\| g[x(t; t_0, x_0)] \| \leqslant b^{-1}(\beta) = \alpha_1$ (say), $(t \geqslant t_0, x_0 \in A)$. Due to (4.6) there exists an $\alpha_2 = \alpha_2(\beta)$ such that

$$\sup_{\| g(x) \| \leqslant \alpha_2} \sum_{i=1}^{N} V_i(t, x) \leqslant \beta.$$

Let

$$\alpha = \min(\alpha_1, \alpha_2). \tag{4.1a}$$

If follows therefore that whenever $x_0 \in A$, $\| g[x(t; t_0, x_0)] \| \leqslant \alpha$. I.e., $x(t; t_0, x_0) \subset \overline{S(A, \alpha)}$ for all $t \geqslant t_0$. Thus $\overline{S(A, \alpha)} = B$ (say) is the conditionally invariant set with respect to A.

Now let $\epsilon > 0$ be given. As (ia) holds, given $b(\alpha + \epsilon) > 0$ and $t_0 > 0$, where α is the same number as defined by (4.1a), there exists a positive function $\delta = \delta(t_0, \epsilon, \alpha)$ that is continuous in t_0 for each ϵ such that

$$\sum_{i=1}^{N} r_i(t; t_0, r_0) < b(\alpha + \epsilon) \qquad (t \geqslant t_0) \tag{4.10}$$

whenever

$$\sum_{i=1}^{N} r_{i0} \leqslant \delta.$$

Now choose r_0 satisfying (4.2) such that

$$\sum_{i=1}^{k} r_{i0} \leqslant \delta, \tag{4.11}$$

and

$$r_{i0} = 0 \qquad (i = k + 1, \cdots, N). \tag{4.12}$$

In view of (4.9) and (3.2), (4.12) $\Rightarrow x_0 \in M_k$ because of (4.4). Further, from the monotonic property of $b(r)$, (4.5), (4.11), (4.12), and (4.9) we deduce that

$$\| g(x_0) \| \leqslant b^{-1}(\delta) = \delta_1 \text{ (say)}.$$

Also because of (4.6) there exists a $\delta_2 = \delta_2(\delta)$ such that

$$\sup_{\| g(x_0) \| \leqslant \delta_2} \sum_{i=1}^{N} V_i(t_0, x_0) \leqslant \delta.$$

Let $\delta_3 = \min(\delta_1, \delta_2)$. Now $\| g(x_0) \| \leqslant \delta_3 \Leftarrow x_0 \in \overline{S(A, \delta_3)}$.

It follows therefore that whenever $x_0 \in \overline{S(A, \delta_3)} \cap M_k$ every solution $x(t; t_0, x_0)$ satisfies (4.8).

Suppose if possible that a solution $x(t; t_0, x_0)$ of (2.1), $x_0 \in \overline{S(A, \delta_3)} \cap M_k$, is such that $\| g[x(t; t_0, x_0)] \| = \alpha + \epsilon$ for some $t = t_1 \geqslant t_0$. Then using the relations (4.5), (4.8), and (4.10) we get the contradiction

$$b(\alpha + \epsilon) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1; t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t_1; t_0, r_0) < b(\alpha + \epsilon)$$

which proves that (ia) $\Rightarrow$ (i). We now proceed to the proof of the assertion that (iiia) $\Rightarrow$ (iii).

Let $\epsilon > 0$, $\gamma > 0$ and $t_0 \geqslant 0$ be given. Let $\| g(x_0) \| \leqslant \gamma$. Then because of (4.6) we can choose a $\gamma_1 = \gamma_1(\gamma)$ such that

$$\sup_{\| g(x_0) \| \leqslant \gamma} \sum_{i=1}^{N} V_i(t_0, x_0) \leqslant \gamma_1 .$$

Since (4.9) $\Rightarrow$ (4.8), we choose r_{i0} $(i = 1, \cdots, N)$ such that (4.12) and $\sum_{i=1}^{k} r_{i0} \leqslant \gamma_1$ hold. As before, one concludes from (4.3), (4.12), and $\| g(x_0) \| \leqslant \gamma$ that whenever $x_0 \in \overline{S(A, \gamma)} \cap M_k$, every solution $x(t; t_0, x_0)$ of (2.1) satisfies (4.8).

Now as (iiia) holds, given $b(\alpha + \epsilon) > 0$, $\gamma_1 > 0$ there exists a positive number $T = T(t_0, \gamma_1, \epsilon)$ such that

$$\sum_{i=1}^{N} r_i(t; t_0, r_0) < b(\alpha + \epsilon) \qquad (t \geqslant t_0 + T) \tag{4.13}$$

whenever

$$\sum_{i=1}^{N} r_{i0} \leqslant \gamma_1 .$$

Let $\{t_n\}$ be a sequence such that $t_n \to \infty$ as $n \to \infty$ and $t_n \geqslant t_0 + T$ for each n. Then the assumption that a solution $x(t; t_0, x_0)$ of (2.1) such that $x_0 \in \overline{S(A, \gamma)} \cap M_k$ has the property that $\| g[x(t_n; t_0, x_0)] \| \geqslant \alpha + \epsilon$ leads to the contradiction

$$b(\alpha + \epsilon) \leqslant \sum_{i=1}^{N} V_i(t_n, x(t_n; t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t_n; t_0, r_0) < b(\alpha + \epsilon)$$

because of the monotonic property of $b(r)$, (4.4), (4.8), and (4.13). This contradiction proves that (iiia) $\Rightarrow$ (iii).

If the conditions (ia) and (iiia) hold simultaneously, then by combining the above proofs, it follows that the condition (va) $\Rightarrow$ (v).

This completes the proof of Theorem 1.

THEOREM 2. *Let the assumptions of the lemma hold together with (4.3), (4.4), (4.5), (4.6), and (4.7). Then*

I. *Condition (iia) $\Rightarrow$ condition (ii),*

II. *Condition (iva) $\Rightarrow$ condition (iv),*

III. *Condition (via) $\Rightarrow$ condition (vi).*

PROOF. The proof follows easily from that of the Theorem 1. For β being independent of t_0, α of (4.1a) is also independent of t_0 as α_1 and α_2 have the same property. Similarly, as (iia) holds, δ is independent of t_0, which implies that δ_3 has the same property. This shows that (iia) $\Rightarrow$ (ii). Similarly for other conditions.

We state below two more theorems. The proofs are omitted as they follow closely the proof of Theorem 1 and with little modification run almost parallel to the corresponding theorem in [3].

THEOREM 3. *Let the conditions of the lemma hold together with (4.3), (4.4), (4.5), (4.6), and (4.7). Then*

I. *Condition (viia) $\Rightarrow$ condition (vii),*

II. *Condition (ixa) $\Rightarrow$ condition (ix),*

III. *Condition (xia) $\Rightarrow$ condition (xi).*

THEOREM 4. *Let the assumptions of the lemma hold together with (4.3), (4.4), (4.5), (4.6). and (4.7). Then*

I. *Condition (viiia) $\Rightarrow$ consition (viii),*

II. *Condition (xa) $\Rightarrow$ condition (x),*

III. *Condition (xiia) $\Rightarrow$ condition (xii).*

We give below an example to illustrate our results with necessary details.

EXAMPLE. Let the differential system be

$$x_1' = (1 + \tfrac{1}{4}\cos t)\, x_1 + (\tfrac{1}{4}\cos t - 1)\, x_2 + e^{(1/4)\sin t - t} \mid x_1 + x_2 \mid^{1/2} + \left[\frac{1 - (t - t_0)(t - t_0 + 1)}{(t - t_0 + 1)^2}\right] \mid x_1 - x_2 \mid^{1/2},$$

$$x_2' = (-1 + \tfrac{1}{4}\cos t)\, x_1 + (\tfrac{1}{4}\cos t + 1)\, x_2 + e^{(1/4)\sin t - t} \mid x_1 + x_2 \mid^{1/2} - \left[\frac{1 - (t - t_0)(t - t_0 + 1)}{(t - t_0 + 1)^2}\right] \mid x_1 - x_2 \mid^{1/2}. \quad (5.1)$$

Let $V = \{V_1, V_2\}$ where

$V_1 = (x_1 + x_2)^2, \quad V_2 = (x_1 - x_2)^2, \quad g(x) = 2(x_1^2 + x_2^2) \quad$ and $\quad b(r) = r$

so that conditions (4.5), (4.6), and (4.7) are satisfied. Let $k = 1$ and by (4.4) $M_1 = \{x_1 : x_1 = x_2\}$. The set A is the set $\{0\}$. Also, condition (4.3) holds. We have

$$V_1^* \leqslant \cos t\, V_1 + 4\, e^{1\backslash 4 \sin t - t}\, V_1^{3\backslash 4}$$

and

$$V_2^* \leqslant 4 V_2 + 4 \left| \frac{1 - (t - t_0)(t - t_0 + 1)}{(t - t_0 + 1)^2} \right| V_2^{3/4}.$$

Thus the function $W = (W_1, W_2)$ takes the form

$$W_1 = \cos t r_1 + 4 e^{(1/4)\sin t - t}\, r_1^{3/4},$$

$$W_2 = 4 r_2 + 4 \left| \frac{1 - (t - t_0)(t - t_0 + 1)}{(t - t_0 + 1)^2} \right| r_2^{3/4}.$$

Clearly the monotonic restrictions on W_1 and W_2 are satisfied. The maxima solution of (3.1) satisfying (4.2) is given by

$$r_1(t; t_0, r_0) = e^{\sin t}[r_{10}^{1/4} e^{-(1/4)\sin t_0} + e^{-t_0} - e^{-t_0}]^4$$

$$r_2(t; t_0, r_0) = \left(\frac{t - t_0}{t - t_0 + 1}\right)^4; \qquad \text{where} \qquad r_0 = (r_{10}, 0). \qquad (5.2)$$

If $r_0 = 0$, we have from (5.2) that

$$\sum_{i=1}^{2} r_i(t; t_0, 0) \leqslant 1 + e^{1-4t_0},$$

so that β of (4.1) is given by $\beta = 1 + e^{1-4t_0}$. It is easy to see that α of (4.1a) is equal to β. Hence the conditionally invariant set with respect to the origin is given by

$$\overline{S(0, \alpha)} = \{x : 2(x_1^2 + x_2^2) \leqslant 1 + e^{1-4t_0}\}.$$

Also the condition (ia) is satisfied. For from (5.2)

$$\sum_{i=1}^{2} r_i(t; t_0, r_0) = \left(\frac{t - t_0}{t - t_0 + 1}\right)^4 + e^{\sin t}[r_{10}^{1/4} e^{-(1/4)\sin t_0} + e^{-t_0} - e^{-t}]^4.$$

Hence given

$$\sum_{i=1}^{2} r_i(t; t_0, r_0) < \beta + \epsilon = 1 + e^{1-4t_0} + \epsilon,$$

we have

$$r_{10} \leqslant \{(1 + e^{4t_0-1}\epsilon)^{1/4} - 1\}^4 e^{-4t_0+\sin t_0}.$$

As $r_{20} = 0$, we have $\delta = r_{10}$. Hence by Theorem 1, condition (i) holds.

NOTE. It is easy to verify that the origin is neither stable nor conditionally stable.

REFERENCES

1. R. BELLMAN. Vector Lyapunov functions. *J. Soc. Ind. Appl. Math.* **A1** (1962), 32-34.
2. V. M. MATROSOV. On the theory of stability of motion (Russian). *Priklad. Mat. Mekhan.* **26** (1962), 992-1002.
3. V. LAKSHMIKANTHAM. Vector Lyapunov functions and conditional stability. *J. Math. Anal. Appl.* **10** (1965), 368-377.
4. T. WAZEWSKI. Systems des équations et des inéqualités différentielles ordinaires aux denxié mes membres monotones et leurs applials. *Ann. Soc. Polon. Math.* **23** (1950), 112-166.